The Substitution Rule

Indefinite integral

$\int f(x)\,dx$ = General antiderivative of $f(x)$

$\int f(x)\,dx = F(x) + C$

particular antiderivative

ie $F'(x) = f(x)$

Recall Chain Rule :

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$$

$\Rightarrow$ $F(g(x))$ = antiderivative of $f(g(x))\,g'(x)$

$$\Rightarrow \boxed{\int f(g(x))\,g'(x)\,dx = F(g(x)) + C}$$

Example $\int (x^2+1)^7 \cdot 2x \; dx = ?$

Let $f(x) = x^7$ and $g(x) = x^2 + 1$.

$\Rightarrow f(g(x))\,g'(x) = (x^2+1)^7 \cdot 2x$

$\int f(x)\,dx = \int x^7\,dx = \frac{1}{8}x^8 + C$

Choose $F(x) = \frac{1}{8}x^8$

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

$$\Rightarrow \int (x^2+1)^7\, 2x \, dx = \frac{1}{8}(x^2+1)^8 + C$$

<u>Potential Problem</u> : It might be hard to spot appropriate $f(x)$ and $g(x)$ at the start.

More methodical approach : <u>Integration by Substitution</u>

1/ Examine integrand. Is it roughly of the form $f(g(x))\,g'(x)$, at least up to a constant.

E.g. $(x^2+1)^7 \cdot 2x$

2/ We're going to change the variable in the antiderivative to make it easier. Set

new variable

$u = g(x)$

relationship between x and u.

E.g. $u = x^2+1$

a) Replace dx with expression involving du as follows :

$$\frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)}$$

Pretend we're solving for dx using algebra. It's weird but its going to be helpful.

E.g. $\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$$\Rightarrow \int (x^2+1)^7 \cdot 2x \, dx = \int (x^2+1)^7 \cdot \cancel{2x} \cdot \frac{du}{\cancel{2x}}$$

$$= \int (x^2+1)^7 \cdot du$$

b) Rewrite integrand purely in terms of the u variable.

E.g. $\int (x^2+1)^7 \cdot du = \int u^7 \, du$

3/ Calculate indefinite integral in the u-variable.

E.g. $\int u^7 \, du = \frac{1}{8} u^8 + C$

4/ Finally switch back to x-variable by replacing u with $g(x)$.

E.g. $\frac{1}{8} u^8 + C = \frac{1}{8}(x^2+1)^8 + C$

Here's why it works in general : $\int f(g(x)) g'(x) \, dx = ?$

$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)}$

$$\Rightarrow \int f(g(x)) g'(x) \, dx = \int f(g(x)) \cancel{g'(x)} \frac{du}{\cancel{g'(x)}}$$

$$= \int f(g(x)) \, du = \int f(u) \, du = F(u) + C$$

$$= F(g(x)) + C$$

(As expected by Chain Rule)

Conclusion :

Hopefully easier to calculate

$$u = g(x) \Rightarrow \int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

Examples

1/ $$\int x^2 \sqrt{x^3+2}\, dx = ?$$

$$u = x^3 + 2 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$$

$$\Rightarrow \int x^2 \sqrt{x^3+2}\, dx = \int x^2 \sqrt{x^3+2} \cdot \frac{du}{3x^2}$$

$$= \int \frac{1}{3} \sqrt{x^3+2}\, du = \int \frac{1}{3} u^{\frac{1}{2}}\, du$$

$$= \frac{1}{3} \cdot \frac{1}{\frac{1}{2}+1} u^{\frac{1}{2}+1} + C = \frac{2}{9} u^{3/2} + C$$

$$= \frac{2}{9} (x^3+2)^{3/2} + C$$

2/ $$\int 2^{(x^2)} \cdot 5x\, dx = ?$$

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x} \Rightarrow$$

$$\int 2^{(x^2)} \cdot 5x\, dx = \int 2^{(x^2)} \cdot 5x \frac{du}{2x}$$

$$= \int \frac{5}{2} \cdot 2^{(x^2)}\, du = \int \frac{5}{2} 2^u\, du$$

$$= \frac{5}{2} \cdot \frac{1}{\ln(2)} 2^u + C = \frac{5}{2} \cdot \frac{1}{\ln(2)} 2^{(x^2)} + C$$

3/ $\displaystyle\int \frac{(\ln(x))^2}{x}\,dx = ?$

$$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

$$\Rightarrow \int \frac{(\ln(x))^2}{x}\,dx = \int \frac{(\ln(x))^2}{\cancel{x}}\,\cancel{x}\,du$$

$$= \int (\ln(x))^2\,du = \int u^2\,du = \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(\ln(x))^3 + C$$

4/ $\displaystyle\int \frac{x+1}{x^2+2x}\,dx = ?$

$$u = x^2 + 2x \Rightarrow \frac{du}{dx} = 2x + 2$$

$$\Rightarrow dx = \frac{du}{2(x+1)}$$

$$\Rightarrow \int \frac{x+1}{x^2+2x}\,dx = \int \frac{\cancel{x+1}}{x^2+2x} \cdot \frac{du}{2(\cancel{x+1})} = \int \frac{1}{2} \cdot \frac{1}{x^2+2x}\,du$$

$$= \int \frac{1}{2} \cdot \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+2x| + C$$

No obvious choice of u

5/ $\displaystyle\int x\sqrt{1+2x}\,dx = ?$

Try $u = 1 + 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

$\left(\Rightarrow x = \frac{u-1}{2}\right)$

$$\Rightarrow \int x\sqrt{1+2x}\, dx = \int x\sqrt{1+2x} \cdot \frac{du}{2}$$

$$= \int \frac{u-1}{2}\sqrt{u} \cdot \frac{1}{2}\, du = \int \frac{1}{4}u^{3/2} - \frac{1}{4}u^{1/2}\, du$$

$$= \frac{1}{4} \cdot \frac{1}{3/2+1} u^{3/2+1} - \frac{1}{4} \cdot \frac{1}{\frac{1}{2}+1} u^{1/2+1} + C$$

$$= \frac{1}{10} u^{5/2} - \frac{1}{6} u^{3/2} + C$$

$$= \frac{1}{10}(1+2x)^{5/2} - \frac{1}{6}(1+2x)^{3/2} + C$$

<u>Major Warning</u>: We won't be able to use this to integrate straight compositions $\int f(g(x))\, dx$. Without $g'(x)$ we'll run into problems.

E.g. $\int e^{(x^2)}\, dx = ?$

$u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$(\Rightarrow x = \sqrt{u})$

$$\Rightarrow \int e^{(x^2)} dx = \int e^{(x^2)} \frac{du}{2x} = \int \frac{e^u}{2\sqrt{u}}\, du$$

More complicated

Can't calculate

Very <u>WRONG</u>: $\frac{d}{dx}\left(e^{(x^2)}\right) = e^{x^2} \cdot 2x$

$$\Rightarrow \frac{1}{2x}\frac{d}{dx}\left(e^{(x^2)}\right) = e^{(x^2)}$$

WRONG. $\frac{1}{2x}$ not a constant. We're forgetting product rule

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{2x} e^{(x^2)}\right) = e^{(x^2)}$$

$$\Rightarrow \int e^{(x^2)}\,dx = \frac{1}{2x} e^{(x^2)} + C$$

Totally incorrect. Differentiate and see what you get. Remember product rule

More generally : $\displaystyle \int f(g(x))\,dx \neq \frac{F(g(x))}{g'(x)} + C$

Conclusion We have no method to calculate $\int f(g(x))\,dx$ in general. We can only deal with integrals of the form $\int f(g(x)) \cdot g'(x)\,dx$.

constants

Special Case : $\displaystyle \int f(ax+b)\,dx = ?$

Try $u = ax+b \Rightarrow \frac{du}{dx} = a \Rightarrow dx = \frac{du}{a} \Rightarrow$

constant

$$\int f(ax+b)\,dx = \int f(ax+b)\,\frac{du}{a} = \int \frac{1}{a} f(u)\,du$$

$$= \frac{1}{a} F(u) + C = \frac{1}{a} F(ax+b) + C$$

Conclusion :

$$\boxed{\int f(x)\,dx = F(x) + C \Rightarrow \int f(ax+b)\,dx = \frac{1}{a} F(ax+b) + C}$$

Example : $\int \frac{1}{3x+4}\,dx = ?$

$$\int \frac{1}{x}\,dx = \ln|x| + C \Rightarrow \int \frac{1}{(3x+4)}\,dx = \frac{1}{3}\ln|3x+4| + C$$

Overview of Integration by Substitution

x-world (Original variable)	$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)}$	u-world (New variable)

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C \xleftarrow{\text{Replace } u \text{ with } g(x)} F(u) + C \xleftarrow{\text{Calculate integral}} \int f(u)\,du$$

Replace dx

$$\int f(g(x))\,\cancel{g'(x)}\,\frac{du}{\cancel{g'(x)}} = \int f(g(x))\,du$$

Replace $g(x)$ with u